M.Math. IInd year — II Semestral exam 2009
Algebraic number theory — B.Sury
Answer any FIVE
Be brief; quote precisely the results you use.

Q 1.
For any Galois extension L/K of number fields, show that the decomposition group at almost all prime ideals of $\mathcal{O}_L$ are cyclic.

OR

Prove that for any Galois extension K of $\mathbf{Q}$, the Galois group is generated by the various inertia subgroups of the primes.

Q 2.
Show that the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is $p^{-1/(p-1)}$ in k where $[k : Q_p] < \infty$.

OR

Determine with proof the unramified extension of a given degree n over $\mathbf{Q}_p$.

Q 3.
For a modulus $\mathcal{M}$ of a number field K, define the ray class group mod $\mathcal{M}$. Show that $I_K^{\mathcal{M}}/i(K_{\mathcal{M}}) \cong C_K$, the class group of K.

OR

Let $f \in \mathcal{O}_K[X]$ be monic, and assume that the Galois group of f over K fixes at at least one root of f. Then, for every non-zero prime ideal P of $\mathcal{O}_K$, the equation $f(X) = 0$ has a solution in $\mathcal{O}_K$ mod P.

Q 4.
State the Frobenius density theorem and use it to deduce that for an abelian extension L/K, the Artin map gives a surjection from $I_K^{\mathcal{M}}$ onto $\mathrm{Gal}(L/K)$, provided $\mathcal{M}$ is divisible by all ramified places.

OR

State the Frobenius density theorem and use it to prove that in a cyclic extension of algebraic number fields, infinitely many primes are inert.
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P.T.O.

Q 5.
Let $G = < \sigma >$ be a cyclic group of order n acting on a free abelian group $A = \sum_{i=1}^{d} Zv_i$ of rank d dividing n in the following manner :

$$\sigma(v_i) = v_{i+1} \ \forall \ i < d \ ; \ \sigma(v_d) = v_1.$$

Compute the Herbrand quotient $q(A)$.

OR

Let L/K be a cyclic extension of algebraic number fields. For a finite place v of L, show that the unit group U_v of L_v is a module for the decomposition group at v and that $H^1(U_v)$ has order e_v.

Q 6.
Let $n!(X^n/n! + X^{n-1}/(n-1)! + \ldots + X + 1) = f(X)g(X)$ with $f, g \in \mathbf{Z}[X]$ where f is irreducible. Prove that any prime p dividing $f(0)$ is $\leq$ deg (f).
Hint: Work in the field $K = Q(\theta)$ where θ is a root of f. Write

$$\theta^n + n\theta^{n-1} + \cdots + n!\theta = -n!$$

and consider the $\mathcal{P}$-adic valuations of both sides for some $\mathcal{P}$ lying over p.

OR

Let $K = Q(\zeta)$, where ζ is a primitive n-th root of unity. If S is the set of prime ideals containing (n) and $I_Q(S)$ denotes the group of fractional ideals generated by primes outside S, find the image and the kernel of the Artin map from $I_Q(S)$ to Gal (K/Q).